MA 125-8B, CALCULUS I

October 16, 2014

Name (Print last name first): Key

Show all your work and justify your answer!

No partial credit will be given for the answer only!

PART I

You must simplify your answer when possible.
All problems in Part I are 10 points each.

1. Find the absolute maximum and minimum of the function $y = f(x) = (x^2 - 1)^3$ on the interval $[-1, 2]$.

Critical Points

$f'(x) = 3(x^2-1)^2(2x)$

$= 6x((x-1)(x+1))^2$

$f'(x) = 0$ when

$x = 0, x = 1, x = -1$

Closed Interval Method

$f(0) = -1$

$f(1) = 0$

$f(-1) = 0$

$f(2) = 27$

Absolute max is 27 at $x = 2$

Absolute min is -1 at $x = 0$

2. Find the number c which satisfies the conclusion of the Mean Value Theorem for the function $y = f(x) = x^2 + x$ on the interval $[0, 4]$.

We want c in $(0,4)$ where $f'(c) = \frac{f(4)-f(0)}{4-0} = \frac{20-0}{4-0} = 5$

$f'(x) = 2x+1$ so $f'(c) = 2c+1 = 5$ gives

$2c = 4$

$c = 2.$

3. Find all critical numbers of the function $y = f(x) = 2x^3 + 3x^2 - 36x + 12$ and identify all local max/min if any.

$f'(x) = 6x^2 + 6x - 36$

$= 6(x^2 + x - 6)$

$= 6(x+3)(x-2)$

Critical points are

$x = 2$ & $x = -3$.

	-4		0		3
		-3		2	
$x+3$	−		+		+
$x-2$	−		−		+
$f'(x)$	+		−		+
		∧		∨	

local max at $x = -3$

local min at $x = 2$

4. Suppose that the **derivative** of a function $y = f(x)$ is:
$f'(x) = x^2 - x - 12$.

(a) Find the x-coordinates of all local max/min of the function $y = f(x)$.

$f'(x) = x^2 - x - 12 = 0$

$(x-4)(x+3) = 0$

critical points $x = 4$
$x = -3$

	-4		0		5
		-3		4	
$x-4$	−		−		+
$x+3$	−		+		+
$f'(x)$	+	∧	−	∨	+

local max at $x = -3$

local min $x = 4$

(b) At which x is the function $y = f(x)$ most rapidly decreasing?

$f''(x) = 2x - 1$

$f''(x) = 0$ when

$x = \frac{1}{2}$

	0		1
		1/2	
$f''(x)$	−		+
		∨	

$x = 1/2$ is an absolute min of $f'(x)$, thus f decreases fastest at $x = \frac{1}{2}$.

PART II

5. **[15 points]** The concentration of an average student during a 3 hour test at time t is given by $C(t) = 2t^3 - 3t^2 - 12t + 20$. When, during the test, is the student's concentration maximal?

Closed interval method since $0 \leq t \leq 3$.

$C'(t) = 6t^2 - 6t - 12 = 6(t^2 - t - 2) = 6(t-2)(t+1)$

critical points at $t = 2$ & $t = -1$, but only $t = 2$ is in our interval.

$C(0) = 20$

$C(3) = 2(3)^3 - 3(3)^2 - 12(3) + 20$

$= 2(27) - 3(9) - 36 + 20$

$= 54 - 27 - 36 + 20$

$= 11$

$C(2) = 2(2)^3 - 3(2)^2 - 12(2) + 20$

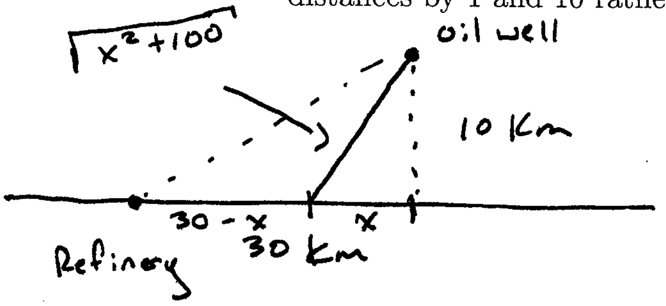
$= 2(8) - 3(4) - 24 + 20$

$= 16 - 12 - 24 + 20$

$= 0$

Absolute max is 20 at $t = 0$.

6. **[15 points]** An oil refinery is located on the shore and an oil well is located 10 km off shore 30 km east of the refinery. [Hence if the refinery is located at (0, 0) and the x-axis is the shore line, then the well is located at (30, 10).] If it costs 10 million per kilometer to lay a pipe line in the ocean and 1 million per kilometer to lay a pipe line on land, how should one lay the pipe line from the well to the refinery to minimize the cost? Tip: To avoid working with very large numbers, you should use cost coefficients in millions per kilometer. That is, multiply your distances by 1 and 10 rather than 1,000,000 and 10,000,000.



Let x be the distance the pipe is not run along the shore. Then $0 \le x \le 30$. Let $C(x)$ be the cost.

Closed interval method!

Then $C(x) = 1(30-x) + 10\sqrt{x^2+100}$.

$$C'(x) = -1 + 10\left(\frac{1}{2}(x^2+100)^{-1/2}(2x)\right)$$

$$= -1 + \frac{10x}{\sqrt{x^2+100}} = 0$$

$$\frac{10x}{\sqrt{x^2+100}} = 1$$

$$10x = \sqrt{x^2+100}$$

$$100x^2 = x^2+100$$

$$99x^2 = 100$$

$$x^2 = \frac{100}{99}$$

$$x = \frac{10}{\sqrt{99}}$$

$$C(0) = 30 + 10\sqrt{100}$$
$$= 130$$

$$C(30) = 10\sqrt{900+100}$$
$$= 10\sqrt{1000}$$
$$= 100\sqrt{10}$$
$$\approx 316$$

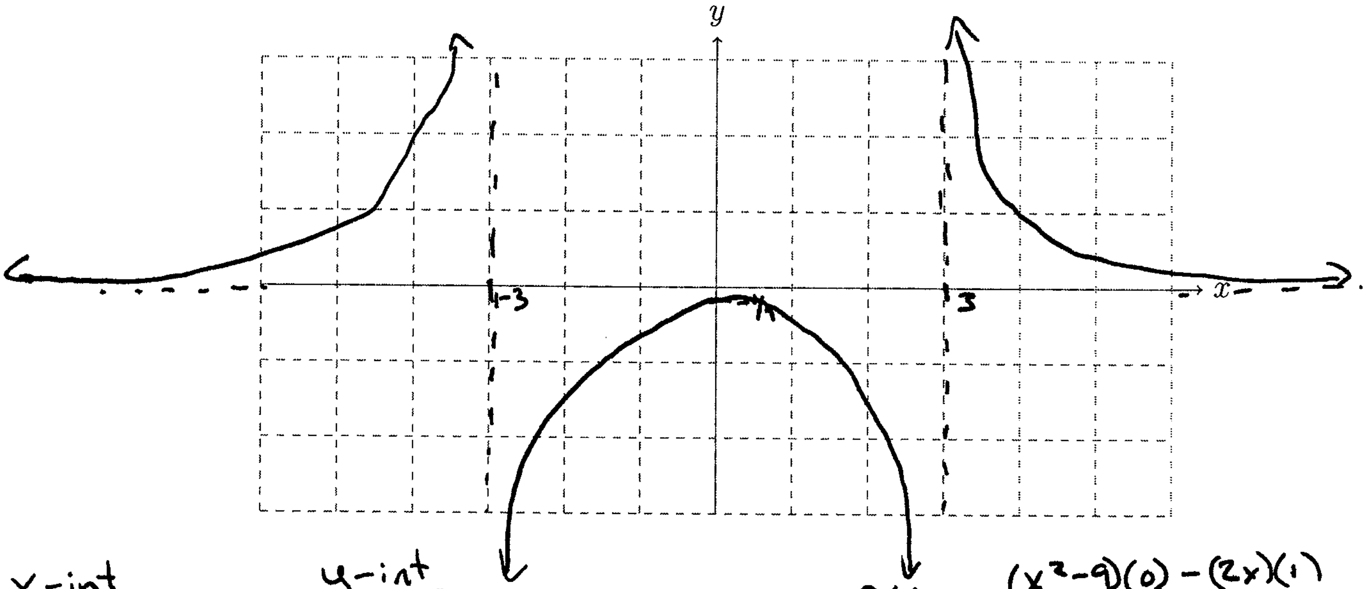
$$C\left(\frac{10}{\sqrt{99}}\right) = 30 - \frac{10}{\sqrt{99}} + 10\sqrt{\frac{100}{99}+100}$$
$$\approx 129.5$$

So the cost is minimized when $x = \frac{10}{\sqrt{99}} \approx 1.005$ Km

7. [**20 points**] **Use calculus** to graph the function $y = f(x) = \frac{1}{x^2 - 9}$. Indicate

- x and y intercepts,
- vertical and horizontal asymptotes (if any),
- in/de-creasing; local max/min (if any).

You must show work to justify your graph and conclusions. You can use decimal numbers to plot points (but mark them with exact values).



x-int

$0 = \frac{1}{x^2-9}$

no x-int

y-int

$y = \frac{1}{0^2-9} = -\frac{1}{9}$

$f'(x) = \frac{(x^2-9)(0)-(2x)(1)}{(x^2-9)^2}$

$= \frac{-2x}{(x^2-9)^2}$

Horizontal Asymptotes

$\lim_{x\to\infty} \frac{1}{x^2-9} = 0$

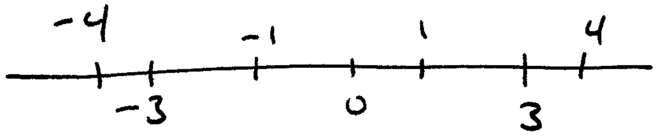
$\lim_{x\to-\infty} \frac{1}{x^2-9} = 0$

Vertical Asymptotes

$\lim_{x\to-3^-} \frac{1}{x^2-9} = \infty$ $\lim_{x\to 3^-} \frac{1}{x^2-9} = -\infty$

$\lim_{x\to-3^+} \frac{1}{x^2-9} = -\infty$ $\lim_{x\to 3^+} \frac{1}{x^2-9} = \infty$

critical pts $x=0$, $x=3$, $x=-3$



$-2x$	+	+	−	−
$f'(x)$	+	+	−	−

Increasing on $(-\infty,-3)\cup(-3,0)$ local max @ $x=0$

Decreasing on ~~$(0,\infty)$~~ $(0,3)\cup(3,\infty)$

8. This question has two parts.

(a) [**6 points**] Show that the equation $y = f(x) = 2x^3 + 3x - \sin(x) + \frac{1}{100} = 0$ has **exactly** one solution.

$f(0) = \frac{1}{100} > 0$

$f(-1) = -2 - 3 - \sin(-1) + \frac{1}{100}$

$= \left(-5 + \frac{1}{100}\right) + \sin(1) < 0$

So, IVT says there is at least one root in $(-1, 0)$.

Suppose there are two roots at $x = a$ & $x = b$. Then Rolle's Theorem says there is a c in (a, b) where $f'(c) = 0$.

$f'(x) = 6x^2 + 3 - \cos(x)$

To find c we solve

$6x^2 + 3 - \cos(x) = 0$

$\Rightarrow \cos(x) = 6x^2 + 3 \geq 3$

so no solution. Thus, f has only one root.

(b) [**4 points**] Find the linearization of $f(x)$ at $a = 0$ and use this linearization to approximate the solution of $f(x) = 0$.

$L(x) = f(a) + f'(a)(x - a)$

$f(0) = \frac{1}{100}$ $\quad f'(0) = 3 - 1 = 2$

$L(x) = \frac{1}{100} + 2x$

To approximate $f(x) = 0$ we solve $L(x) = 0$.

$L(x) = \frac{1}{100} + 2x = 0$

$2x = -\frac{1}{100}$

$$\boxed{x = \frac{-1}{200}}$$