MA 125-8B, CALCULUS I

Test 3, November 6, 2014

Name (Print last name first): ...Key...

Show all your work and justify your answer!

No partial credit will be given for the answer only!

PART I

You must simplify your answer when possible.
All problems in Part I are 7 points each.

1. If $F(x) = \int_1^x \cos(t^3)\,dt$, find the derivative $F'(x)$.

By the Fundamental Theorem of Calculus,

$$F'(x) = \cos(x^3).$$

2. Use a Riemann sum with $n = 3$ terms and the right endpoint rule to approximate $\int_1^2 \cos(\frac{1}{x})\,dx$. (You don't need to multiply or add the terms.)

$\Delta x = \frac{2-1}{3} = \frac{1}{3}$

$x_0 = 1$

$x_1 = 1 + 1/3 = \frac{4}{3}$

$x_2 = 1 + 2/3 = \frac{5}{3}$

$x_3 = 2$

$$\int_1^2 \cos\left(\frac{1}{x}\right)dx \approx f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x$$

$$= \cos\left(\frac{3}{4}\right)\left(\frac{1}{3}\right) + \cos\left(\frac{3}{5}\right)\left(\frac{1}{3}\right) + \cos\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)$$

3. Evaluate $\int x(x^2+3)\,dx$

$$\int x(x^2+3)dx = \int x^3+3x \; dx$$

$$= \frac{1}{4}x^4 + \frac{3}{2}x^2 + C$$

4. Evaluate $\int x\cos(x^2)\,dx$

Let $u = x^2$, then $du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$.

$$\int \cos(x^2)(x\,dx) = \int \cos(u)\left(\frac{1}{2}du\right)$$

$$= \frac{1}{2}\int \cos(u)\,du$$

$$= \frac{1}{2}\sin(u) + C$$

$$= \frac{1}{2}\sin(x^2) + C$$

5. Evaluate $\int \frac{x^2+2}{\sqrt{x}}\,dx$

$$\int \frac{x^2+2}{\sqrt{x}}dx = \int x^{3/2} + 2x^{-1/2} \, dx$$

$$= \frac{x^{5/2}}{5/2} + 2\left(\frac{x^{1/2}}{1/2}\right) + C$$

$$= \frac{2}{5}x^{5/2} + 4x^{1/2} + C$$

6. Find the average value of the function $f(x) = x^3$ on the interval $[0, 3]$.

$$f_{ave} = \frac{1}{3-0} \int_0^3 x^3\,dx$$

$$= \frac{1}{3}\left(\frac{1}{4}x^4 \Big|_0^3\right.$$

$$= \frac{1}{3}\left(\frac{1}{4}(3)^4 - 0\right)$$

$$= \frac{81}{12} = \frac{27}{4}$$

7. Evaluate $\int \cos^2(x)\sin(x)\,dx$.

Let $u = \cos(x)$.

$du = -\sin(x)\,dx$

$-du = \sin(x)\,dx$

$$\int \cos^2(x)\sin(x)\,dx = \int u^2(-du)$$

$$= -\int u^2\,du$$

$$= -\frac{1}{3}u^3 + C$$

$$= -\frac{1}{3}\cos^3(x) + C$$

8. Evaluate $\displaystyle\int_{-2}^{2} \frac{x^2\sin(x)}{x^4+5}\,dx$

$$\int_{-2}^{2} \frac{x^2\sin(x)}{x^4+5}\,dx = 0$$ since $\dfrac{x^2\sin(x)}{x^4+5}$ is an odd function.

PART II

All problems in Part II are 11 points each.

1. Evaluate $\int_0^1 (x^2+1)\sqrt{x^3+3x}\,dx$

$$\int_0^1 (x^2+1)\sqrt{x^3+3x}\,dx = \int_0^4 \sqrt{u}\left(\tfrac{1}{3}du\right)$$

$$= \frac{1}{3}\int_0^4 u^{1/2}\,du$$

$$= \frac{1}{3}\left(\frac{2}{3}u^{3/2}\right|_0^4$$

$$= \frac{1}{3}\left(\frac{2}{3}(4)^{3/2} - 0\right)$$

$$= \frac{1}{3}\left(\frac{2}{3}\right)(8) = \frac{16}{9}$$

Let $u = x^3+3x$

$du = 3x^2+3\,dx$

$\frac{1}{3}du = (x^2+1)dx$

when $x=0$,

$u=0$

when $x=1$

$u=(1)^3+3(1)=4$

2. Evaluate $\displaystyle\int \frac{x^3}{\sqrt{x^2+1}}\,dx$

$$\int \frac{x^3}{\sqrt{x^2+1}}\,dx = \int \frac{x^2}{\sqrt{x^2+1}}\,x\,dx$$

$$= \int \frac{u-1}{\sqrt{u}}\left(\tfrac{1}{2}du\right)$$

$$= \frac{1}{2}\int u^{1/2} - u^{-1/2}\,du$$

$$= \frac{1}{2}\left(\frac{2}{3}u^{3/2} - 2u^{1/2} + C\right)$$

$$= \frac{1}{3}(x^2+1)^{3/2} - (x^2+1)^{1/2} + C$$

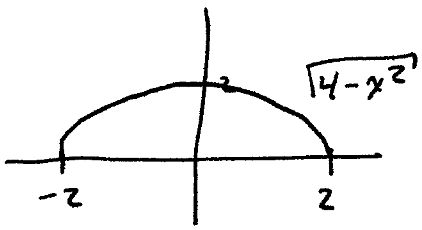
Let $u = x^2+1$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$x^2 = u-1$

3. Evaluate $\int_{-2}^{2} \sqrt{4-x^2}\,dx$. (Hint: Consider the graph.)



Area of a semicircle of radius 2 is $\frac{1}{2}\pi(2)^2 = 2\pi$. Thus,

$$\int_{-2}^{2} \sqrt{4-x^2}\,dx = 2\pi$$

4. A rocket is launched vertically from ground level. The velocity of the rocket is given by $v(t) = 45t^2\,\frac{m}{s}$. How far does the rocket travel during the first eight seconds after launch?

$$\int_0^8 v(t)\,dt = \int_0^8 45t^2\,dt$$

$$= 45\int_0^8 t^2\,dt$$

$$= 45\left(\frac{1}{3}t^3\Big|_0^8\right.$$

$$= 45\left(\frac{1}{3}(8)^3 - 0\right)$$

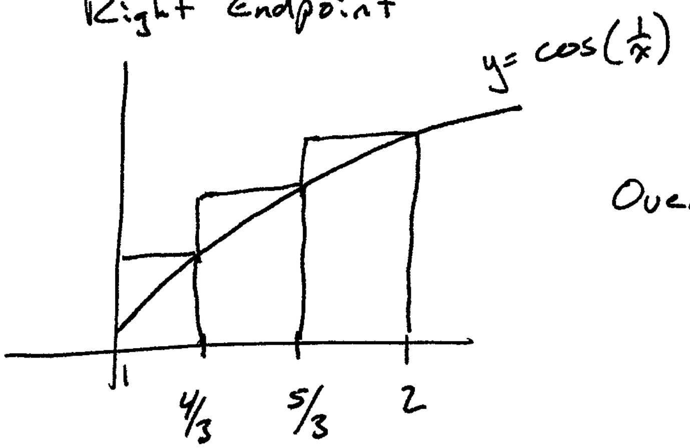
$$= 45\left(\frac{512}{3}\right)$$

$$= \frac{23040}{3} = 7680\text{ m}$$

Bonus question worth at most 5 points!

5. In part I, problem 2 you used a Riemann sum with $n = 3$ terms and the right endpoint rule to approximate $\int_1^2 \cos(\frac{1}{x})\,dx$. Is this estimate an upper or lower estimate? As always, you must justify your answer. Find another estimate (with three terms) so that the true value of the integral is in between the two estimates. What should you use as your best approximate value and what is its error at most? (You don't need to add and multiply the terms.)

Right Endpoint

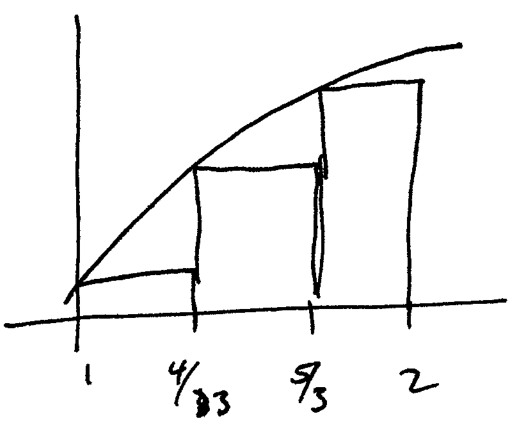


Overestimate

$\int_1^2 \cos(\frac{1}{x})dx$ is between these two values.

Best approximation should be the average of the two. Error is at most the difference of the two estimates.

Letf Endpoint



Underestimate

$$f(1)\left(\tfrac{1}{3}\right) + f\left(\tfrac{4}{3}\right)\left(\tfrac{1}{3}\right) + f\left(\tfrac{5}{3}\right)\left(\tfrac{1}{3}\right)$$

$$= \cos(1)\left(\tfrac{1}{3}\right) + \cos\left(\tfrac{3}{4}\right)\left(\tfrac{1}{3}\right) + \cos\left(\tfrac{3}{5}\right)\left(\tfrac{1}{3}\right)$$