Trigonometric Formulas and Identities for MA106

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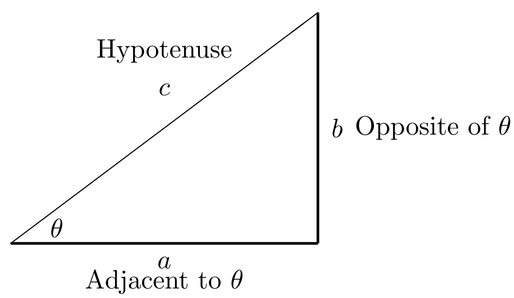
April 15, 2013

1 Trigonometric Functions of Acute Angles



Function Name	Abbreviation	Value	Sides
sine of θ	$\sin\theta$	$\frac{b}{c}$	$\frac{\text{opposite}}{\text{hypontenuse}}$
cosine of θ	$\cos\theta$	$\frac{a}{c}$	$\frac{\text{adjacent}}{\text{hypontenuse}}$
tangent of θ	$\tan\theta$	$\frac{b}{a}$	$\frac{\text{opposite}}{\text{adjacent}}$
cosecant of θ	$\csc\theta$	$\frac{c}{b}$	$\frac{\text{hypontenuse}}{\text{opposite}}$
secant of θ	$\sec\theta$	$\frac{c}{a}$	$\frac{\text{hypontenuse}}{\text{adjacent}}$
cotangent of θ	$\cot\theta$	$\frac{a}{b}$	$\frac{\text{adjacent}}{\text{opposite}}$

2 Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

3 Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad\qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

4 Pythagorean Identities

The Pythagorean Theorem gives the following relation between the edges of a right triangle. For a right triangle with hypotenuse c and legs a and b,

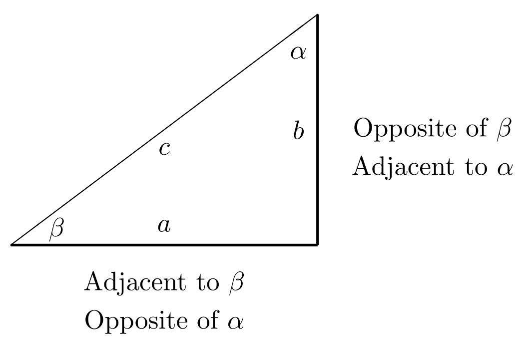
$$a^2 + b^2 = c^2.$$

The following identities can be shown directly from the Pythagorean Theorem.

$$\sin^2\theta + \cos^2\theta = 1 \qquad\qquad \frac{\sin^2\theta}{\cos^2\theta} + 1 = \frac{1}{\cos^2\theta}$$

$$\tan^2\theta + 1 = \sec^2\theta \qquad\qquad \cot^2\theta + 1 = \csc^2\theta$$

5 Complementary Angle Theorem



$$\sin\beta = \frac{b}{c} = \cos\alpha \qquad \cos\beta = \frac{a}{c} = \sin\alpha \qquad \tan\beta = \frac{b}{a} = \cot\alpha$$

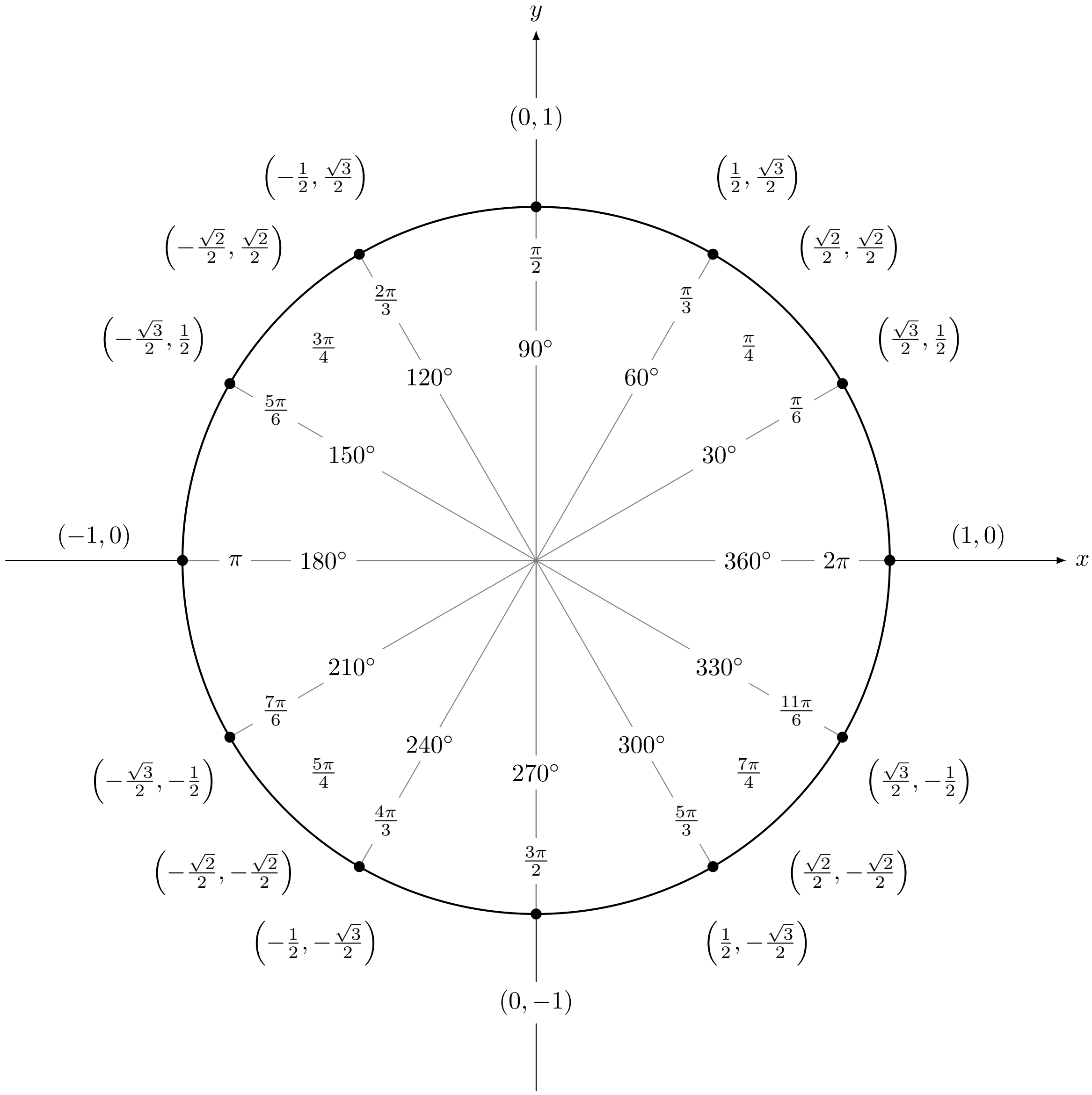
$$\csc\beta = \frac{c}{b} = \sec\alpha \qquad \sec\beta = \frac{c}{a} = \csc\alpha \qquad \cot\beta = \frac{a}{b} = \tan\alpha$$

θ(Degrees)	θ(Radians)
$\sin\theta = \cos(90^\circ - \theta)$	$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$
$\cos\theta = \sin(90^\circ - \theta)$	$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$
$\tan\theta = \cot(90^\circ - \theta)$	$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$
$\csc\theta = \sec(90^\circ - \theta)$	$\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$
$\sec\theta = \csc(90^\circ - \theta)$	$\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$
$\cot\theta = \tan(90^\circ - \theta)$	$\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$

6 Special Angles

θ(Degrees)	θ(Radians)	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
0	0°	0	1	0	Not defined	1	Not defined
$\frac{\pi}{6}$	30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$\frac{\pi}{4}$	45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\frac{\pi}{3}$	60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{2}$	90°	1	0	Not defined	1	Not defined	0
π	180°	0	-1	0	Not defined	-1	Not defined
$\frac{3\pi}{2}$	270°	-1	0	Not defined	-1	Not defined	0

7 Sine and Cosine on a Unit Circle



8 Determining Sign by Quadrant

Quadrant of θ	$\sin\theta$, $\csc\theta$	$\cos\theta$, $\sec\theta$	$\tan\theta$, $\cot\theta$
I	Positive	Positive	Positive
II	Positive	Negative	Negative
III	Negative	Negative	Positive
IV	Negative	Positive	Negative

9 Reference Angles

If θ is an angle that lies in a quadrant and if α is its reference angle, then

$$\sin\theta = \pm\sin\alpha \quad \cos\theta = \pm\cos\alpha \quad \tan\theta = \pm\tan\alpha$$

$$\csc\theta = \pm\csc\alpha \quad \sec\theta = \pm\sec\alpha \quad \cot\theta = \pm\cot\alpha$$

where the $+$ or $-$ sign depends on the quadrant in which θ lies.

10 Trigonometric Functions on a Circle

For an angle θ in standard position, let the point $P = (a, b)$ lie on the terminal side of θ as well as the circle $x^2 + y^2 = r^2$, then

$$\sin\theta = \tfrac{b}{r} \qquad \cos\theta = \tfrac{a}{r} \qquad \tan\theta = \tfrac{b}{a},\ a \neq 0$$

$$\csc\theta = \tfrac{r}{b},\ b \neq 0 \quad \sec\theta = \tfrac{r}{a},\ a \neq 0 \quad \cot\theta = \tfrac{a}{b},\ b \neq 0$$

11 Periodic Properties of Trigonometric Functions

$$\sin(\theta + 2\pi) = \sin\theta \quad \cos(\theta + 2\pi) = \cos\theta \quad \tan(\theta + 2\pi) = \tan\theta$$

$$\csc(\theta + 2\pi) = \csc\theta \quad \sec(\theta + 2\pi) = \sec\theta \quad \cot(\theta + 2\pi) = \cot\theta$$

12 Even-Odd Properties of Trigonometric Functions

$$\sin(-\theta) = -\sin\theta \quad \cos(-\theta) = \cos\theta \quad \tan(-\theta) = -\tan\theta$$

$$\csc(-\theta) = -\csc\theta \quad \sec(-\theta) = \sec\theta \quad \cot(-\theta) = -\cot\theta$$

13 Properties of the Trigonometric Functions

Properties of the Sine Function $y = \sin x$

1. The domain is the set of all real numbers.
2. The range consists of all real numbers from -1 to 1, inclusive.
3. The sine function is an odd function.
4. The sine function is periodic, with period 2π.
5. The x-intercepts are $\ldots - 2\pi, -\pi, 0, \pi, 2\pi, \ldots$; the y-intercept is 0.

6. The maximum value is 1 and it occurs at $x = \ldots, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \ldots$; the minimum is -1 and it occurs at $x = \ldots, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots$.

Properties of the Cosine Function $y = \cos x$

1. The domain is the set of all real numbers.
2. The range consists of all real numbers from -1 to 1, inclusive.
3. The cosine function is an even function.
4. The cosine function is periodic, with period 2π.
5. The x-intercepts are $\ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$; the y-intercept is 1.
6. The maximum value is 1 and it occurs at $x = \ldots, -2\pi, 0, 2\pi, \ldots$; the minimum is -1 and it occurs at $x = \ldots, -\pi, \pi, 3\pi, \ldots$.

Properties of the Tangent Function $y = \tan x$

1. The domain is the set of all real numbers, except odd multiples of $\frac{\pi}{2}$.
2. The range is the set of all real numbers.
3. The tangent function is an odd function.
4. The tangent function is periodic, with period π.
5. The x-intercepts are $\ldots - 2\pi, -\pi, 0, \pi, 2\pi, \ldots$; the y-intercept is 0.
6. Vertical asymptotes occur at $x = \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$.

14 Amplitude and Period

If $\omega > 0$, the amplitude, period, and phase shift of $y = A\sin(\omega x - \phi)$ and $y = A\cos(\omega x - \phi)$ are

$$\text{Amplitude} = |A| \qquad \text{Period} = T = \frac{2\pi}{\omega} \qquad \text{Phase Shift} = \frac{\phi}{\omega}.$$

The phase shift if to the left if $\phi < 0$ and to the right if $\phi > 0$.

15 Inverse Trigonometric Functions

$$\begin{aligned}
\sin^{-1}(\sin x) &= x, && \text{where } -\tfrac{\pi}{2} \le x \le \tfrac{\pi}{2} \\
\sin(\sin^{-1} x) &= x, && \text{where } -1 \le x \le 1 \\
\cos^{-1}(\cos x) &= x, && \text{where } 0 \le x \le \pi \\
\cos(\cos^{-1} x) &= x, && \text{where } -1 \le x \le 1 \\
\tan^{-1}(\tan x) &= x, && \text{where } -\tfrac{\pi}{2} < x < \tfrac{\pi}{2} \\
\tan(\tan^{-1} x) &= x, && \text{where } -\infty < x < \infty
\end{aligned}$$

16 Finding All Solutions to Trigonometric Equations

n	$\sin x = n$	$\cos x = n$
$\|n\| < 1$	$x = \alpha + 2k\pi$ $x = \pi - \alpha + 2k\pi$ $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$	$x = \pm\alpha + 2k\pi$ $\alpha \in (0, \pi)$
$n = -1$	$x = -\frac{\pi}{2} + 2k\pi$	$\pi + 2k\pi$
$n = 0$	$x = k\pi$	$x = \frac{\pi}{2} + k\pi$
$n = 1$	$x = \frac{\pi}{2} + 2k\pi$	$x = 2k\pi$
$\|n\| > 1$	No solution	No solution

n	$\tan x = n$
General Case	$x = \alpha + k\pi$ $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$
$n = -1$ $n = 0$ $n = 1$	$x = -\frac{\pi}{4} + k\pi$ $x = k\pi$ $x = \frac{\pi}{4} + k\pi$

17 Sum and Difference Formulas

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

18 Double-Angle Formulas

$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$
$$\cos(2\theta) = 1 - 2\sin^2\theta$$
$$\cos(2\theta) = 2\cos^2\theta - 1$$
$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$$

19 Half-Angle Formulas

$$\sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2} \qquad \sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$
$$\cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2} \qquad \cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$
$$\tan^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{1+\cos\alpha} \qquad \tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$
$$\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha} \qquad \tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha}$$

20 Product-to-Sum Formulas

$$\sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta)-\cos(\alpha+\beta))$$
$$\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha-\beta)+\cos(\alpha+\beta))$$
$$\sin\alpha\cos\beta = \frac{1}{2}(\sin(\alpha+\beta)+\sin(\alpha-\beta))$$

21 Sum-to-Product Formulas

$$\sin\alpha+\sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$
$$\sin\alpha-\sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$$
$$\cos\alpha+\cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$
$$\cos\alpha-\cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

22 Law of Sines

Theorem 22.1 (Law of Sines)**.** *For a triangle with sides a, b, c and opposite angles A, B, C, respectively,*

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

23 Law of Cosines

Theorem 23.1 (Law of Cosines)**.** *For a triangle with sides a, b, c and opposite angles A, B, C, respectively,*

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$a^2 = b^2 + c^2 - 2bc\cos A$$

24 Heron's Formula

The area K of a triangle with sides a, b, and c is

$$K = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a+b+c)$.

25 Converting Polar Coordinates to Rectangular Coordinates

If P is a point with polar coordinates (r, θ), the rectangular coordinates (x, y) of P are given by

$$x = r\cos\theta \qquad\qquad y = r\sin\theta.$$

26 Converting Rectangular Coordinates to Polar Coordinates

For $a > 0$, if the point (x, y) lies on one of the rectangular coordinate axes, then

$$(x,y) = (a,0) \longrightarrow (r,\theta) = (a,0)$$
$$(x,y) = (0,a) \longrightarrow (r,\theta) = \left(a, \frac{\pi}{2}\right)$$
$$(x,y) = (-a,0) \longrightarrow (r,\theta) = (a,\pi)$$
$$(x,y) = (0,-a) \longrightarrow (r,\theta) = \left(a, \frac{3\pi}{2}\right).$$

If the point (x, y) does not lie on one of the rectangular coordinate axes, then

$$r = \sqrt{x^2 + y^2}.$$

If (x, y) lies in Quadrant I or IV, then

$$\theta = \tan^{-1}\frac{y}{x}.$$

If (x, y) lies in Quadrant II or III, then

$$\theta = \pi + \tan^{-1}\frac{y}{x}.$$

27 Parabolas

27.1 Parabolas with vertex as $(0,0)$

Vertex at $(0,0)$; Focus on an axis; $a>0$

Vertex	Focus	Directrix	Equation	Description
$(0,0)$	$(a,0)$	$x=-a$	$y^2=4ax$	Opens right
$(0,0)$	$(-a,0)$	$x=a$	$y^2=-4ax$	Opens left
$(0,0)$	$(0,a)$	$y=-a$	$x^2=4ay$	Opens up
$(0,0)$	$(0,-a)$	$y=a$	$x^2=-4ay$	Opens down

27.2 Parabolas with vertex at (h,k)

Vertex at (h,k); $a>0$

Vertex	Focus	Directrix	Equation	Description
(h,k)	$(h+a,k)$	$x=h-a$	$(y-k)^2=4a(x-h)$	Opens right
(h,k)	$(h-a,k)$	$x=h+a$	$(y-k)^2=-4a(x-h)$	Opens left
(h,k)	$(h,k+a)$	$y=k-a$	$(x-h)^2=4a(y-k)$	Opens up
(h,k)	$(h,k-a)$	$y=k+a$	$(x-h)^2=-4a(y-k)$	Opens down

28 Ellipses

28.1 Ellipses with center at $(0,0)$

Center	Major Axis	Foci	Vertices	Equation
$(0,0)$	x-axis	$(-c,0)$	$(-a,0)$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$
		$(c,0)$	$(a,0)$	$a>b$ and $b^2=a^2-c^2$
$(0,0)$	y-axis	$(0,-c)$	$(0,-a)$	$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1,$
		$(0,c)$	$(0,a)$	$a>b$ and $b^2=a^2-c^2$

28.2 Ellipses with center at (h,k)

Center	Major Axis	Foci	Vertices	Equation
(h,k)	Parallel to x-axis	$(h-c,k)$	$(h-a,k)$	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1,$
		$(h+c,k)$	$(h+a,k)$	$a>b$ and $b^2=a^2-c^2$
(h,k)	Parallel to y-axis	$(h,k-c)$	$(h,k-a)$	$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1,$
		$(h,k+c)$	$(h,k+a)$	$a>b$ and $b^2=a^2-c^2$

29 Hyperbolas

29.1 Hyperbolas with center at $(0, 0)$

Center	**Transverse Axis**	**Foci**	**Vertices**	**Equation**	**Asymptotes**
$(0, 0)$	x-axis	$(-c, 0)$	$(-a, 0)$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$	$y = \frac{b}{a}x$
		$(c, 0)$	$(a, 0)$	$b^2 = c^2 - a^2$	$y = -\frac{b}{a}x$
$(0, 0)$	y-axis	$(0, -c)$	$(0, -a)$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1,$	$y = \frac{a}{b}x$
		$(0, c)$	$(0, a)$	$b^2 = c^2 - a^2$	$y = -\frac{a}{b}x$

29.2 Hyperbolas with center at (h, k)

Center	**Transverse Axis**	**Foci**	**Vertices**	**Equation**	**Asymptotes**
(h, k)	Parallel to x-axis	$(h - c, k)$	$(h - a, k)$	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1,$	$y - k = \frac{b}{a}(x - h)$
		$(h + c, k)$	$(h + a, k)$	$b^2 = c^2 - a^2$	$y - k = -\frac{b}{a}(x - h)$
(h, k)	Parallel to y-axis	$(h, k - c)$	$(h, k - a)$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1,$	$y - k = \frac{a}{b}(x - h)$
		$(h, k + c)$	$(h, k + a)$	$b^2 = c^2 - a^2$	$y - k = -\frac{a}{b}(x - h)$